

# F-theory compactifications on manifolds with $SU(3)$ structure

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## Abstract

In this paper we derive part of the low energy action corresponding to F-theory compactifications on specific eight manifolds with  $SU(3)$  structure. The setup we use can actually be reduced to compactification of six-dimensional supergravity coupled to tensor multiplets on a  $T^2$  with duality twists. The resulting theory is a  $N = 2$  gauged supergravity coupled to vector-tensor multiplets.

# 1 Introduction

Recently it was pointed out that in the presence of certain fluxes, the heterotic –type IIA duality in four dimensions requires that, on the type IIA side, M-theory has to be considered instead. The fluxes which are responsible for this behaviour are ordinary fluxes for the heterotic gauge fields [1]. The full duality picture is heterotic string compactified on  $K3 \times T^2$  with duality twists is the same as M-theory compactified on seven-dimensional manifolds with  $SU(3)$  structure which are obtained by fibering Calabi–Yau manifolds over a circle [2].

It turns out that the heterotic picture can be further generalised by allowing twists in the full 4d-duality group [3]. This construction gives what is known under the name of R-fluxes [4]. It has been conjectured that the dual of this setup can be found in F-theory compactifications on eight-dimensional manifolds with  $SU(3)$  structure obtained by fibering a Calabi–Yau manifold over a  $T^2$  much in the same way as it was done in the M-theory case [2]. Motivated by this, we study the tensor multiplet sector of such F-theory compactifications. This leads to  $N = 2$  supergravity theories in four dimensions coupled to vector-tensor multiplets.

## 2 General setup

We are interested in F-theory compactifications on eight-dimensional manifolds with  $SU(3)$  structure obtained by fibering a Calabi–Yau manifold over a two torus,  $T^2$ . The fibration is done such that the two-forms on the Calabi–Yau manifold satisfy

$$d\omega_\alpha = -M_i^\beta{}_\alpha \omega_\beta \wedge dz^i . \quad (2.1)$$

where  $i, j = 1, 2$  denote the torus directions, while  $\omega_\alpha$  denote the harmonic two-forms on the Calabi–Yau manifold and the matrices  $M_1$  and  $M_2$  are constant commuting matrices which are in the algebra of the symmetry group on the space of two-forms.

Since there is no low energy effective action description for F-theory, a direct compactification is not possible and we have to rely on other methods. In particular for the case above, the fibration can be effectively realised by splitting the compactification into a compactification on a Calabi–Yau three-fold followed by a Scherk–Schwarz compactification [5] on the torus. After the first step, the the six-dimensional fields which come from an expansion in the forms  $\omega_\alpha$  which satisfy (2.1) would have a non-trivial dependence on the torus coordinates, which is why one has to consider a Scherk–Schwarz compactification in order to obtain the correct result.

Let us specify more the compactification Ansatz. We consider throughout that the Calabi–Yau three-fold is elliptically fibered with four-dimensional base  $\mathcal{B}$ . The two-forms may have two origins: two forms which come from the base of the fibration and two-forms which come from resolving the singularities of the fibration. In the following we shall concentrate only on the first type of two-forms, namely the ones which already exist on the base of the fibration. It is known that in F-theory compactifications on Calabi–Yau 3-folds these forms lead in six dimensions to antisymmetric tensor fields. It is precisely this tensor-field/ tensor-multiplet sector that will be of interest for us in the following.

If we denote the number of  $(1,1)$  forms on  $\mathcal{B}$  by  $h^{1,1}(\mathcal{B})$  then,  $T$ , the number of tensor multiplets is given by  $T = h^{1,1}(\mathcal{B}) - 1$ . Note that supersymmetry requires that

$h^{2,0}(\mathcal{B}) = 0$ , and therefore, all the two forms of interest – and in particular the forms in (2.1) – are the  $(1, 1)$  forms on  $\mathcal{B}$ . On such a four-dimensional space there is precisely one self-dual  $(1, 1)$  form (the Kähler form) and  $T$  anti-self-dual  $(1, 1)$  forms. This implies that the inner product on the space of two-forms possesses a  $SO(1, T)$  symmetry. This symmetry is nothing but the symmetry found in [6] on the space of tensor-fields in six-dimensional  $N = 1$  supergravity coupled to  $T$  tensor multiplets. Therefore, we choose the twist matrices  $M_1$  and  $M_2$  to be generators of  $SO(1, T)$ .

Let us summarize. We have just argued that F-theory compactifications on eight-dimensional manifolds obtained by fibering a Calabi–Yau manifold over a torus as described in (2.1) can be effectively modeled by considering six dimensional compactifications of F-theory followed by a compactification on a torus with  $SO(1, T)$  duality twists. In particular we shall be interested in tensor-multiplet sector of the six-dimensional theory.

### 3 Compactification with duality twists

#### 3.1 The six-dimensional theory

Let us start by describing the content of the theory in six dimensions. A similar description of the theory appeared recently in [7]. We are interested in six-dimensional minimal supergravity coupled to  $T$  tensor multiplets. We suppose throughout that the number of hypermultiplets is such that the gravitational anomalies are canceled. The supergravity multiplet contains as bosonic degrees of freedom the graviton  $g_{\mu\nu}$  and an antisymmetric tensor field with self-dual field strength. Each of the tensor multiplets contain as bosonic degrees of freedom one antisymmetric tensor field with anti-self-dual field strength and one scalar field. The (anti-)self-duality of these tensor fields can also be seen from the F-theory/type IIB compactification. Recall that type IIB string features in ten dimensions a RR four-form potential,  $C_4$ , with self-dual field strength. When expanded in the  $(1, 1)$  harmonic forms on the base  $\mathcal{B}$  of the Calabi–Yau three-fold this precisely yields one tensor field with self-dual field strength and  $h^{1,1}(\mathcal{B}) - 1 \equiv T$  tensor fields with anti-self-dual field strengths.

Let us denote all the tensor fields generically by  $B^\alpha, \alpha = 1, \dots, T + 1$  and the Kähler moduli corresponding to deformations of the base by  $v^\alpha$ . These fields appear from the expansion of the RR four-form  $C_4$  and of the Kähler form  $J$  in a basis of  $(1, 1)$  harmonic forms on the base  $\mathcal{B}$ .

$$C_4 = \dots + B^\alpha \omega_\alpha + \dots ; \quad J = v^\alpha \omega_\alpha . \quad (3.1)$$

Note that we work with a basis of  $(1, 1)$  forms in which the (anti) self-duality is not manifest. Let us define the intersection numbers on  $\mathcal{B}$  by

$$\rho_{\alpha\beta} = \int_B \omega_\alpha \wedge \omega_\beta . \quad (3.2)$$

The matrix  $\rho$  has  $(1, T)$  signature and is the matrix which is used to raise and lower  $SO(1, T)$  indices. The volume of the base which is defined as

$$\mathcal{V} = \frac{1}{2} \int_B J \wedge J = \frac{1}{2} \rho_{\alpha\beta} v^\alpha v^\beta , \quad (3.3)$$

is part of a hypermultiplet. Therefore in order to correctly describe the number of  $T$  scalar degrees of freedom by  $T + 1$  variables  $v^\alpha$  we shall work at constant volume,  $\mathcal{V} = 1$ .

It has been known from [8, 6] that these theories admit a manifestly Lorenz invariant Lagrangean description only in the case  $T = 1$ . For an arbitrary number of tensor-multiplets – and here we want to keep this number arbitrary – the self-duality conditions make it impossible to derive the theory from an action principle. However, since we are only interested in the four-dimensional compactified theory, we shall adopt a strategy, which was used in type IIB compactifications [9], which will allow us to circumvent the above problem. The idea is to write down an action for tensor fields whose field strengths are not constrained by any self-duality condition. In this way we double the number of degrees of freedom described by the tensor fields. After the compactification to four dimensions the additional degrees of freedom manifest themselves as independent fields which are Poincare dual to the normal degrees of freedom which we would have expected from the compactification. By adding suitable Lagrange multiplier terms to the action we can impose the four-dimensional version of the self-duality conditions as the equations of motion for the additional degrees of freedom in the theory. Eliminating at this step these degrees of freedom from their equations of motion we obtain the theory we were searching in the first place.

Therefore we consider the following starting six-dimensional action<sup>1</sup>

$$S = -\frac{1}{2} \int \left( R + \frac{1}{2} g_{\alpha\beta} \hat{H}^\alpha \wedge * \hat{H}^\beta + g_{\alpha\beta} d\hat{v}^\alpha \wedge * d\hat{v}^\beta \Big|_{\mathcal{V}=1} \right), \quad \alpha, \beta = 1, \dots, T+1, \quad (3.4)$$

where  $\hat{H}^\alpha$  denotes the field strength for the tensor fields which is given by

$$\hat{H}^\alpha = d\hat{B}^\alpha. \quad (3.5)$$

The metric  $g_{\alpha\beta}$  can be seen as coming from the F-theory/type IIB compactification as<sup>2</sup>

$$g_{\alpha\beta} = \int_{\mathcal{B}} \omega_\alpha \wedge * \omega_\beta, \quad (3.6)$$

and has a  $SO(1, T)$  isometry group. In order to have the correct theory we have to impose the self-duality conditions

$$* \hat{H}^\alpha = \rho^{\alpha\beta} g_{\beta\gamma} \hat{H}^\gamma, \quad (3.7)$$

by hand as they can not be derived from the action (3.4). This relation is self-consistent precisely due to the  $SO(1, T)$  symmetry which ensures that

$$g^{-1}{}^{\alpha\beta} = \rho^{\alpha\gamma} g_{\gamma\delta} \rho^{\delta\beta}. \quad (3.8)$$

The above data specify the six-dimensional action. We shall use this formulation in the next section in order to perform a compactification on a torus with duality twists.

### 3.2 Scherk-Schwarz compactification to four dimensions

In this section we perform the Scherk–Schwarz toroidal compactification of the six-dimensional theory presented before. Let us start by describing the degrees of freedom we expect in the

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<sup>1</sup>Hats  $\hat{\phantom{x}}$  are used in order to distinguish six-dimensional fields from their four-dimensional descendants.

<sup>2</sup>Up to factors of  $\mathcal{V}$  which are irrelevant as we set  $\mathcal{V} = 1$ .

four-dimensional theory. From the gravity sector there will be two Kaluza–Klein vector fields  $V^{1,2}$  and three torus moduli which we shall take as the three independent components of the metric on the torus  $G^{11}$ ,  $G^{12}$  and  $G^{22}$ . One of the vector fields will be the graviphoton, the scalar superpartner of the graviton in four dimensions, while the other vector field together with two of the torus moduli will become the bosonic components of a vector multiplet.

From the tensor fields compactified on the torus we expect the following degrees of freedom

$$\hat{B}^\alpha = B^\alpha + A_i^\alpha \wedge dz^i + b^\alpha dz^1 \wedge dz^2 . \quad (3.9)$$

Due to the (anti)self-duality condition which the corresponding field-strengths satisfy the number of degrees of freedom is only half of the ones above. In particular we expect one vector field and either a scalar field or a tensor field. In case we keep the scalar we will end up with a true vector multiplet while if we keep the tensor we will have a vector-tensor multiplet. The additional scalars in these multiplets are given by the remaining torus modulus above and the scalars which already exist in six dimensions as superpartners of the tensor fields. Altogether we will end up with a number of  $T + 2$  vector plus vector-tensor multiplets.

Let us now see how the compactification proceeds. As explained in the previous section, the part of the theory we are interested in has a  $SO(1, T)$  duality symmetry. We shall use this symmetry in order to perform the compactification with duality twists. In particular we are interested in the following dependencies on the internal coordinates of the torus

$$\begin{aligned} \partial_i \hat{B}^\alpha &= M_i^\alpha{}_\beta \hat{B}^\beta , \\ \partial_i \hat{v}^\alpha &= M_i^\alpha{}_\beta \hat{v}^\beta . \end{aligned} \quad (3.10)$$

The sign difference compared to (2.1) comes from the fact that we are adopting the passive rather than the active picture for the symmetry transformations. Note that the  $SO(1, T)$  transformation of the volume of the base  $\mathcal{B}$  is given by

$$\delta \mathcal{V} = \rho_{\alpha\beta} M_i^\alpha{}_\gamma \hat{v}^\gamma \hat{v}^\beta + \rho_{\alpha\beta} \hat{v}^\alpha M_i^\beta{}_\gamma \hat{v}^\gamma , \quad (3.11)$$

which vanishes because the generators  $(M_i)_{\alpha\beta} = \rho_{\alpha\gamma} M_i^\gamma{}_\beta$  are antisymmetric in the indices  $\alpha$  and  $\beta$ , ie

$$\rho_{\alpha\gamma} M_i^\gamma{}_\beta + \rho_{\beta\gamma} M_i^\gamma{}_\alpha = 0 . \quad (3.12)$$

Note that this is consistent with the fact that the volume of the base is part of a hypermultiplet.

Let us consider the standard metric for the compactification on  $T^2$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + G_{ij} (dz^i - V^i) (dz^j - V^j) , \quad (3.13)$$

where  $g_{\mu\nu}$  is the metric on the four-dimensional space,  $G_{ij}$  is the metric on the torus and by  $V^i$ ,  $i = 1, 2$  we denoted the Kaluza–Klein vector fields which come from the torus compactification.

The field strengths  $\hat{H}^\alpha$  which come from the expansion (3.9) read

$$\hat{H}^\alpha = dB^\alpha + (dA_i^\alpha + M_i^\alpha{}_\beta B^\beta) \wedge dz^i + \left( db^\alpha + M_{2\beta}^\alpha A_1^\beta - M_{1\beta}^\alpha A_2^\beta \right) dz^1 \wedge dz^2 , \quad (3.14)$$

while for the scalar fields  $v^\alpha$  we find

$$d\hat{v}^\alpha = dv^\alpha + M_i^\alpha v^\beta dz^i . \quad (3.15)$$

Note that in toroidal compactifications the basis for field expansions  $dz^i$  is not invariant under rezidual diffeomorphism transformations – which induce four-dimensional gauge transformations – and therefore the fields which result from this expansion will have non-standard transformation properties. This has also a rather technical consequence. Since Hodge  $*$  operator in (3.4) is taken with respect to the metric (3.13) which is non-diagonal, it will introduce crossed terms between the four-dimensional space and the torus. A factorization can still be achieved if we use for the expansion of the fields involved in (3.4) the gauge invariant basis

$$\eta^i = dz^i - V^i . \quad (3.16)$$

This is precisely the basis which should be used in order to obtain fields with correct gauge transformations. Rewriting the field strengths (3.14) and (3.15) in this basis we obtain

$$\begin{aligned} \hat{H}^\alpha &= H^\alpha + F_i^\alpha \wedge \eta^i + Db^\alpha \eta^1 \wedge \eta^2 , \\ d\hat{v}^\alpha &= Dv^\alpha - M_i^\alpha v^\beta \eta^i , \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} H^\alpha &= dB^\alpha + F_1 \wedge V^1 + F_2 \wedge V^2 - Db^\alpha \wedge V^1 \wedge V^2 ; \\ F_1^\alpha &= dA_1^\alpha + M_{1\beta}^\alpha B^\beta - Db^\alpha \wedge V^2 ; \quad F_2^\alpha = dA_2^\alpha + M_{2\beta}^\alpha B^\beta + Db^\alpha \wedge V^1 ; \\ Db^\alpha &= db^\alpha - M_{1\beta}^\alpha A_2^\beta + M_{2\beta}^\alpha A_1^\beta ; \quad Dv^\alpha = dv^\alpha + M_i^\alpha v^\beta V^i . \end{aligned} \quad (3.18)$$

For the forms on the torus we use the following normalisation  $\int_{T^2} \eta^1 \wedge \eta^2 = 1$ , which implies

$$\int_{T^2} \eta^i \wedge * \eta^j = \sqrt{G} G^{ij} , \quad \int_{T^2} *1 = G \int_{T^2} \eta^1 \wedge \eta^2 \wedge *(\eta^1 \wedge \eta^2) = \sqrt{G} . \quad (3.19)$$

Performing the integration over the torus the tensor field part in the action (3.20) becomes

$$S_T = -\frac{1}{4} \int \sqrt{G} \left( g_{\alpha\beta} H^\alpha \wedge *H^\beta + g_{\alpha\beta} G^{ij} F_i^\alpha \wedge *F_j^\beta + \frac{1}{G} g_{\alpha\beta} Db^\alpha \wedge *Db^\beta \right) . \quad (3.20)$$

To this we have to add the part of the action which descends from the six-dimensional Ricci scalar and from the kinetic term of the scalars  $v^\alpha$

$$S_R = -\frac{1}{2} \int \sqrt{G} (R + G_{ij} dV^i \wedge *dV^j + dG_{ij} \wedge *dG^{ij} + g_{\alpha\beta} Dv^\alpha \wedge *Dv^\beta + V) , \quad (3.21)$$

The potential  $V$  comes from the second term in the expansion of  $d\hat{v}^\alpha$  in (3.17) and is given by

$$V = G^{ij} g_{\alpha\beta} M_i^\alpha M_j^\beta v^\delta v^\gamma . \quad (3.22)$$

The correct four-dimensional theory is obtained only after imposing the self-duality conditions (3.7). Inserting the expansion (3.17) into (3.7) we obtain their four-dimensional analogues

$$\begin{aligned} \rho_{\alpha\beta} Db^\beta &= \sqrt{G} g_{\alpha\beta} * H^\beta ; \\ \rho_{\alpha\beta} * F_i^\beta &= \sqrt{G} \epsilon_{ij} G^{jk} g_{\alpha\beta} F_k^\beta . \end{aligned} \quad (3.23)$$

We see that these self-duality conditions identify a field with its Poincare dual, as also explained at the beginning of this section. Evaluating the second relation above for explicit values of the indices  $i, j = 1, 2$ , we obtain

$$\begin{aligned}\frac{1}{\sqrt{G}} * F_1^\alpha &= G^{21} \rho^{\alpha\beta} g_{\beta\gamma} F_1^\gamma + G^{2,2} \rho^{\alpha\beta} g_{\beta\gamma} F_2^\gamma, \\ \frac{1}{\sqrt{G}} * F_2^\alpha &= -G^{11} \rho^{\alpha\beta} g_{\beta\gamma} F_1^\gamma - G^{1,2} \rho^{\alpha\beta} g_{\beta\gamma} F_2^\gamma,\end{aligned}\tag{3.24}$$

which can be easily checked that are equivalent.

### 3.3 Gauge transformations

It will be instructive to collect the gauge transformations for the fields in four dimensions. There are two types of symmetries that we can find. First of all there is the gauge symmetry associated with the tensor fields in six dimensions  $\delta \hat{B}^\alpha = d\hat{\Lambda}^\alpha$  for some gauge parameters  $\hat{\Lambda}^\alpha$  which are one-forms. For these gauge parameters we have to consider a dependence on the torus coordinates which similar to (3.10). The easiest way to see this is to consider the compactification on the full eight-dimensional manifold described by (2.1). Then the gauge invariance above can be obtained by recalling the origin (3.1) of the tensor fields and using the gauge invariance of the four-form  $C_4$

$$\delta C_4 = d\Lambda_3 = d(\Lambda^\alpha \wedge \omega_\alpha + \lambda_i^\alpha \omega_\alpha \wedge dz^i),\tag{3.25}$$

where  $\Lambda^\alpha$  is a 1-form while  $\lambda_i^\alpha$  are scalar functions. Using this, equation (2.1) and the expansion (3.9) we can directly read off the transformations of the four-dimensional fields

$$\delta B^\alpha = d\Lambda^\alpha, \quad \delta A_i^\alpha = -M_i^\alpha{}_\beta \Lambda^\beta; \tag{3.26}$$

$$\delta A_i^\alpha = d\lambda_i^\alpha, \quad \delta b^\alpha = M_1^\alpha{}_\beta \lambda_2^\beta - M_2^\alpha{}_\beta \lambda_1^\beta. \tag{3.27}$$

It is obvious that  $Db^\alpha$  is invariant under the gauge transformation (3.27), while invariance under (3.26) is guaranteed by the fact that the matrices  $M_i$  commute. With this remark it is clear that the other field strengths,  $H^\alpha$  and  $F_{1,2}^\alpha$  are also invariant under (3.26) and (3.27).

The second gauge symmetry we discuss originates from the residual diffeomorphism invariance on the torus. Under the infinitesimal transformation  $\delta z^i = \epsilon^i$ , the KK gauge fields change as  $\delta V^i = d\epsilon^i$  leaving the one-forms  $\eta^i$ , (3.16), invariant. The fields from (3.9) however do not have good transformation properties. In order to obtain fields whose transformations do not involve derivatives of the gauge parameters  $\epsilon_i$  we make the following redefinitions

$$\begin{aligned}A_1^\alpha &= \tilde{A}_1^\alpha + b^\alpha V^2, & A_2^\alpha &= \tilde{A}_2^\alpha - b^\alpha V^1, \\ B^\alpha &= \tilde{B}^\alpha - \tilde{A}_i \wedge V^i + b^\alpha V^1 \wedge V^2.\end{aligned}\tag{3.28}$$

Note that these definitions for the four-dimensional fields can be obtained directly by expanding the six-dimensional fields  $\hat{B}^\alpha$  in the basis  $\eta^i$ . With these definitions we obtain

$$\begin{aligned}H^\alpha &= d\tilde{B}^\alpha + \tilde{A}_i^\alpha \wedge dV^i + M_i^\alpha{}_\beta \tilde{B}^\beta \wedge V^i; \\ F_1^\alpha &= \tilde{F}_1^\alpha + b^\alpha dV^2, & F_2^\alpha &= \tilde{F}_2^\alpha - b^\alpha dV^1 \\ Db^\alpha &= db^\alpha - M_1^\alpha{}_\beta \tilde{A}_2^\beta + M_2^\alpha{}_\beta \tilde{A}_1^\beta + M_i^\alpha{}_\beta b^\beta V^i\end{aligned}\tag{3.29}$$

where  $\tilde{F}_1^\alpha$  and  $\tilde{F}_2^\alpha$  are defined as

$$\tilde{F}_i^\alpha = d\tilde{A}_i^\alpha - M_{j\beta}^\alpha \tilde{A}_i^\beta \wedge V^j - M_{i\beta}^\alpha \tilde{B}^\beta . \quad (3.30)$$

For these new fields the gauge transformations read

$$\delta V^i = d\epsilon^i ; \quad (3.31)$$

$$\delta b^\alpha = -M_{i\beta}^\alpha b^\beta \epsilon^i ; \quad \delta \tilde{A}_i^\alpha = -M_{j\beta}^\alpha \tilde{A}_i^\beta \epsilon^j ; \quad \delta \tilde{B}^\alpha = -M_{i\beta}^\alpha \tilde{B}^\beta \epsilon^i . \quad (3.32)$$

The field strengths transform covariantly

$$\delta D b^\alpha = -M_{i\beta}^\alpha D b^\beta \epsilon^i ; \quad \delta \tilde{F}_i^\alpha = -M_{j\beta}^\alpha \tilde{F}_i^\beta \epsilon^j ; \quad \delta \tilde{H}^\alpha = -M_{i\beta}^\alpha \tilde{H}^\beta \epsilon^i , \quad (3.33)$$

and in order to show this we needed again the fact that the matrices  $M_1$  and  $M_2$  commute.

### 3.4 Imposing the self-duality conditions

The final step, in order to obtain the final four-dimensional action would be to eliminate the doubled degrees of freedom. Note that we shall use the definitions (3.18) and ignore for the moment (3.28). Let us see first which are the fields we would like to keep in the final theory. Regarding the gauge fields it should not be important whether we keep  $A_1$  or  $A_2$  as they appear in a rather symmetric fashion. Suppose we keep  $A_1$ . There is no reason apriori to consider some linear combination of  $A_1$  and  $A_2$ . Recall that in general flux compactifications the kinetic terms are not modified compared to the usual massless compactifications. This is the case for the gauge fields  $A_1$  or  $A_2$ . A linear combination of the gauge fields would make sense if the coefficients are related to the twist matrices so that other parts of the action may be simplified. However such a field redefinition would introduce the twist parameters in the kinetic terms and would put the action in a non-standard form. Shortly we shall motivate on other grounds a twist-dependent redefinition of the gauge fields.

Now let us consider the fields  $B^\alpha$  and  $b^\alpha$ . From the form of the field strengths (3.18) it is clear that the tensor fields are massive due to the Stuckelberg couplings to the vector fields [10, 11, 12]. This means that we have to keep the tensor fields and eliminate the scalars  $b^\alpha$ . Trying to remove the tensor fields from the spectrum would result into scalars which are both electrically and magnetically charged, as it can be seen from their covariant derivative.

The strategy, in order to write the action in terms of  $A_1^\alpha$  and  $B^\alpha$ , is to add suitable total derivative terms to the action such that the variation with respect to  $F_2^\alpha$  and  $D b^\alpha$  reproduces the self-duality constraints. Elimination of the fields  $F_2^\alpha$  and  $D b^\alpha$  from the action would then give the desired result. The terms we will add are of the form  $\rho_{\alpha\beta} d B^\alpha \wedge d b^\beta$  and  $\rho_{\alpha\beta} d A_1^\alpha \wedge d A_2^\beta$  and in order to obtain the self-duality relations we would like to express these terms in terms of the field-strengths  $F_i^\alpha$ ,  $H^\alpha$  and  $D b^\alpha$ . It is straightforward to check that

$$\begin{aligned} S_d &= \rho_{\alpha\beta} H^\alpha \wedge D b^\beta + \rho_{\alpha\beta} F_1^\alpha \wedge F_2^\beta - 2\rho_{\alpha\beta} M_{2\gamma}^\beta d A_1^\alpha \wedge B^\gamma - \rho_{\alpha\beta} M_{1\delta}^\alpha M_{2\gamma}^\beta B^\delta \wedge B^\gamma \\ &= \rho_{\alpha\beta} d B^\alpha \wedge d b^\beta + \rho_{\alpha\beta} d A_1^\alpha \wedge d A_2^\beta - \rho_{\alpha\beta} M_{2\gamma}^\beta d(A_1^\alpha \wedge B^\gamma) + \rho_{\alpha\beta} M_{1\gamma}^\beta d(A_2^\alpha \wedge B^\gamma) , \end{aligned}$$

and therefore  $S_d$  is a total derivative. Let us now consider

$$S_{Total} = S_T - \frac{1}{2} S_d . \quad (3.34)$$



Taking variations of this total action with respect to  $Db^\alpha$  and  $F_2^\alpha$  reproduces the self-duality constraints (3.23). Replacing these constraints into the total action, we see that  $S_T$  identically vanishes as it should have already happened in six dimensions had we imposed the self-duality constraints (3.7) in the action (3.4). Therefore, the only piece we have to deal with is  $S_d$ . This becomes

$$S_d = -\sqrt{G}g_{\alpha\beta}H^\alpha \wedge *H^\beta - \frac{1}{G^{22}}g_{\alpha\beta}F_1^\alpha \wedge *F_1^\beta + \frac{G^{12}}{G^{22}}\rho_{\alpha\beta}F_1^\alpha \wedge F_1^\beta \\ - 2\rho_{\alpha\beta}M_2^\beta{}_\gamma dA_1^\alpha \wedge B^\gamma - \rho_{\alpha\beta}M_1^\alpha{}_\delta M_2^\beta{}_\gamma B^\delta \wedge B^\gamma . \quad (3.35)$$

As anticipated, we end up with a theory for tensor fields which acquire a mass via the Stuckelberg mechanism

$$\delta B^\alpha = d\Lambda^\alpha ; \quad \delta A_1^\alpha = -M_1^\alpha{}_\beta \Lambda^\beta , \quad (3.36)$$

where  $\Lambda^\alpha$  are 1-form gauge parameters. However in trying to replace the self-duality conditions (3.23) in the field strengths (3.18) we obtain cyclic definitions for  $H^\alpha$ . This situation resembles somewhat the results in [13] where it was found that in  $N = 2$  supergravity coupled to vector-tensor multiplets the Bianchi identities require to introduce magnetic dual degrees of freedom. We may try to fix this problem by implementing the redefinitions (3.28) and (3.30). However in this way the fields  $b^\alpha$  will appear in the action without derivative and their replacement using the self-duality conditions (3.23) will no longer be possible. In fact, this can be seen in a more clear way by considering a completely massless compactification where both twist matrices  $M_1$  and  $M_2$  vanish. Even in this case the elimination of the scalars  $b^\alpha$  in the favour of the tensor fields  $B^\alpha$  can not be done consistently. The gauge fields  $A_1$  still have to be redefined according to (3.28). In this way, the scalars  $b^\alpha$  appear in the gauge coupling matrix (not only in the generalised  $\theta$  angles) which proves they are not axions and so we can not expect to be able to dualise them to tensor fields in the usual way. Therefore, we can argue that we have to keep the scalar fields in the resulting theory. However, as we have explained before, in the case that both twist matrices are non-vanishing, the tensor fields are massive and we should rather keep them and not the scalar fields. The way out from this puzzle is to find a different symplectic gauge for the gauge fields where the tensor fields are not explicitly massive and where one can safely remove them from the spectrum. One obvious choice would be to consider as the electric gauge fields the combination which appears in the covariant derivative  $Db^\alpha$  in (3.18). Let us define

$$A_-^\alpha = M_1^\alpha{}_\beta A_2^\beta - M_2^\alpha{}_\beta A_1^\beta . \quad (3.37)$$

Note that, in the corresponding field strength, the tensor fields appear as  $[M_1, M_2]^\alpha{}_\beta B^\beta$  which vanishes due to the fact that the matrices  $M_1$  and  $M_2$  commute. Therefore,  $A_-^\alpha$  are suitable candidates for electric gauge fields. This analysis can be carried out in full generality, but in order to point out the main features we shall choose a particular case,  $M_1 = M_2 = M$ , which is technically less involved. In this particular case we can redefine the gauge fields as

$$A_\pm = A_1 \pm A_2 . \quad (3.38)$$

The field strengths (3.18) become

$$\begin{aligned}
H^\alpha &= dB^\alpha + F_+^\alpha \wedge V^+ + F_-^\alpha \wedge V^- + Db^\alpha \wedge V^+ \wedge V^- ; \\
F_+^\alpha &= dA_+^\alpha + 2M^\alpha{}_\beta B^\beta + Db^\alpha \wedge V^- ; \\
F_-^\alpha &= dA_-^\alpha - Db^\alpha \wedge V^+ ; \\
Db^\alpha &= db^\alpha + M^\alpha{}_\beta A_-^\beta , \quad Dv^\alpha = dv^\alpha + M^\alpha{}_\beta v^\beta V^+ ;
\end{aligned} \tag{3.39}$$

where  $V^\pm = V^1 \pm V^2$ . The action has precisely the same form as (3.20) where now the indices  $i, j$  are understood to take the values  $\pm$  and the metric is given by

$$G_{ij} = \frac{1}{4} \begin{pmatrix} G^{11} + 2G^{12} + G^{22} & G^{11} - G^{22} \\ G^{11} - G^{22} & G^{11} - 2G^{12} + G^{22} \end{pmatrix} \tag{3.40}$$

The field strengths above suggest that it should be possible to keep the gauge fields  $A_-^\alpha$  together with the scalars  $b^\alpha$  and eliminate  $H^\alpha$  and  $F_+^\alpha$ . As before we add a total derivative

$$\begin{aligned}
S_d &= \rho_{\alpha\beta} H^\alpha \wedge Db^\beta - \frac{1}{2} \rho_{\alpha\beta} F_+^\alpha \wedge F_-^\beta - \rho_{\alpha\beta} dA_-^\alpha \wedge V^- \wedge Db^\beta \\
&= \rho_{\alpha\beta} (dB^\alpha \wedge db^\beta - \frac{1}{2} dA_+ \wedge dA_-) - \rho_{\alpha\beta} M^\beta{}_c x d(B^\alpha \wedge A_-^\gamma)
\end{aligned} \tag{3.41}$$

and one can again check that the self-duality conditions written in the  $\pm$  basis can be obtained by taking variations of the total action with respect to  $F_+^\alpha$  and  $H^\alpha$ . Replacing the self-duality conditions in  $S_d$  we obtain

$$\begin{aligned}
S_d &= -\frac{1}{\sqrt{G}} g_{\alpha\beta} Db^\alpha \wedge *Db^\beta - \frac{1}{G^{++}\sqrt{G}} g_{\alpha\beta} F_-^\alpha \wedge *F_-^\beta \\
&\quad + \frac{1}{2} \rho_{\alpha\beta} \frac{G^{+-}}{G^{++}} F_-^\alpha \wedge F_-^\beta - \rho_{\alpha\beta} dA_-^\alpha \wedge V^- \wedge Db^\beta .
\end{aligned} \tag{3.42}$$

With a little bit of effort, the action above can be put in the standard  $N = 2$  gauged supergravity form [14]. We shall not do it explicitly, but we shall just describe the steps which are rather standard. First of all one redefines the gauge fields as  $A_-^\alpha \rightarrow A_-^\alpha + b^\alpha V^+$ . Then the last term in the action above can be integrated by parts

$$\rho_{\alpha\beta} dA_-^\alpha \wedge V^- \wedge Db^\beta = \rho_{\alpha\beta} b^\beta dA_-^\alpha \wedge dV^- + \frac{1}{2} M_{\alpha\beta} A_-^\alpha \wedge A_-^\beta \wedge dV^- + \text{total derivative} . \tag{3.43}$$

One can therefore dualize the gauge field  $V^-$  to its magnetic dual  $\tilde{V}^-$  whose field strength will be of the form  $d\tilde{V}^- + \frac{1}{2} M_{\alpha\beta} A_-^\alpha \wedge A_-^\beta$ . Finally we have to go to the Einstein frame in the action (3.21) and redefine the fields  $v^\alpha$  as

$$v^\alpha = \frac{1}{\sqrt{G}} \tilde{v}^\alpha . \tag{3.44}$$

This effectively means that one of the  $T^2$  moduli, namely  $\sqrt{G}$  becomes part of the scalars  $\tilde{v}^\alpha$  which will no longer be constrained. We can now write the combination  $t^\alpha = b^\alpha + iv^\alpha$  which will have the kinetic term  $g_{\alpha\beta} Dt^\alpha \wedge *Dt^\beta$  where the covariant derivatives are given by

$$Dt^\alpha = dt^\alpha + M^\alpha{}_\beta t^\beta V^+ + M^\alpha{}_\beta A_-^\beta . \tag{3.45}$$

To conclude this section we mention that in the action (3.42) the kinetic terms for the gauge fields depend explicitly on the choice of fluxes  $M_i$ . This may not be completely clear due to the choice we made –  $M_1 = M_2$  – in writing the action (3.42). It is clear however that in the general case this action will not look so simple and moreover the twist matrices will appear explicitly in the kinetic terms for the gauge fields.

### 3.5 Conclusions

In this note we derived part of the action which comes from the compactification of F-theory on certain manifolds with  $SU(3)$  structure. We argued that the compactification can be reduced to a Scherk-Schwarz compactification of six-dimensional supergravity. The direct result is a  $N = 2$  gauged supergravity coupled to vector-tensor multiplets and we have seen that in such a case one can not completely remove the magnetic dual degrees of freedom from the action which is in agreement with the results found in [13]. In a suitable chosen basis for the gauge fields, the magnetic dual degrees of freedom can be decoupled and we end up with ordinary  $N = 2$  gauged supergravity. However, from a physical perspective, The first formulation in terms of vector-tensor multiplets might be more sensible as the usual supergravity quantities (gauge coupling functions in particular) are just given in terms of the geometric data of the compactification manifold as it is the case in massless compactifications which is not the case with the action (3.42). The same point of view may be sustained from the string duality perspective as the dualities are first established at the massless level and only afterwards are deformed to accommodate fluxes. On the other hand we are not aware of any string compactification where vector-tensor multiplets appear non-trivially and therefore the analysis in this paper opens the quest for other compactifications which involve vector-tensor multiplets.

**Acknowledgments** This work was supported in part by the National University Research Council CNCSIS-UEFISCSU, project number PN II-RU 77/04.08.2010 and PN II-ID 464/15.01.2009 and in part by project "Nucleu" PN 09 37 01 02 and PN 09 37 01 06.

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